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Urban income and city size: Ecological Inference with Entropy Econometrics for the Spanish municipalities

Esteban Fernández-Vázquez (evazquez@uniovi.es)*

Fernando Rubiera-Morollón (frubiera@uniovi.es)

Elizabeth Aponte-Jaramillo (University Autonoma of Occidente -Colombia)

University of Oviedo, Department of Applied Economics, Faculty of Economics, Campus del Cristo, Oviedo, 33006, Spain.

* Corresponding author.

Abstract:

Most regional empirical analyses are limited by the lack of data. Researches have to use information that is structured in administrative or political regions which are not always economically meaningful. Any aggregation of a territory in regions in an economic sense requires data on the main economic variables at the level of basic spatial units, such as GDP at a local level. A methodology is proposed in this paper to approximate local GDP values using entropy econometrics which can be defined as an exercise of ecological inference. In addition to the analysis of the main characteristics of the proposed techniques, the paper illustrates how the procedure works taking as an empirical application the estimation of income for Spanish municipalities according to their size. As an example of the possibilities opened up by this methodology, a regional classification based on the relevance of city size, which allows us to measure the relevance of agglomeration economics, is empirically applied to the Spanish case obtaining some interesting first results.

Keywords: urban size and income relationship; entropy econometrics, ecological inference and Spain.

JEL Codes: C15, C21, R11 and R12.

1. Introduction: Why is it Relevant to Obtain Local Data?

With some relevant exceptions, such as the USA, data at a local level are not normally available. The most important economic data, like GDP, are usually presented according to the administrative or political divisions of national territories with not very extensive spatial desegregation. If we use this information, the empirical possibilities of our studies are clearly limited. However, if we are able to use a proper delimitation of *Regions*, with an economic meaning taking in account a particular theoretical framework, specific objective or hypothesis contrast, the analysis will be clear and we shall be able to obtain more relevant conclusions (Behrens and Thisse, 2007).

For instance, one of the most relevant concepts from the urban economic point of view are *agglomeration economies* and *diseconomies* and their effect on location decisions, economic structure and growth between larger and smaller cities (Henderson, J.V. and Thisse, J.F., 2004). *Agglomeration economies* could be approximated by the *size* of the main urban area of a particular region. As XXX shows, a certain degree of international regularity exist linking the *size* of the main city with the average GDP *per capita* in the area. However, these regularities were tested only for some countries and for some specific cities in which local information is available. In the majority of cases, there is not sufficient desegregated information to conduct this type of empirical research.

Another relevant, more classical concept in regional studies is the importance of *distance* and how transportation costs can affect business localization. Through these microeconomic decisions, the macroeconomic structure could be transformed and hence the levels of GDP *per capita* could change. In the literature, all these ideas are typically tested with employment data, which are usually available at a local level. However, the final test with respect to the changes in income linked to the position of each area is not possible in most cases due to this general lack of information at a local level.

We can likewise raise other questions similar to these, such as: How we can evaluate the impact of regional policy at a local level? How we can test the relevance of a new infrastructure? How we can compare different cities and the economic efficiency of the different models of urban growth? The list of relevant questions could be considerably extended while always having to address the basic problem in the majority of cases of lack of data at a local level, especially GDP.

The objective of this paper is to develop a useful approach to obtain this information based on entropy econometrics. The technique could provide us with information at a local level organized according to the size of the main urban area, which is especially interesting from the point of view of most of the models, and for the analysis of regional and urban economies. Our aim is to propose a procedure that could be applied in different scenarios with minor adaptations. In this first step, however, we test the possibilities of the approach by applying it to the Spanish case.

The paper is divided into three further sections. The next section discusses the entropy econometrics solution to an ecological inference problem and presents our methodological proposal. Section 3 presents an empirical application to Spain for 2001 and discusses the results obtained on applying a typical meaningful set of *Regions*. The main conclusions and possible further lines of research complete the paper. Additionally, an appendix reports the outcomes of a Monte Carlo experiment that tests the reliability of the empirical results.

2. The methodology: Ecological Inference with Entropy Econometrics.

2.1. The Maximum Entropy (ME) and Cross Entropy (CE) solutions to pure inverse problems.

In this section, the basics of Entropy Econometrics will be introduced for estimate unknown probabilities in the context of *pure inverse problems*. More extensive introductions can be found in Kapur and Kesavan (1992), Golan *et al*. (1996) or, much more recently, Golan (2006).

Traditionally, probability has been used as a measure of the uncertainty about an event. Let us assume that this event can take *K* possible outcomes $E_1,E_2,...,E_K$ with the respective distribution of probabilities $\boldsymbol{p} = [p_1, p_2, .., p_K]$ such that $\sum_{i=1}^{K} p_i = 1$. Following the formulation of Shannon (1948), the entropy of this distribution $\boldsymbol{p_x}$will be:

$$H(\boldsymbol{p}) = -\sum_{i=1}^{K} p_i ln p_i \qquad (1)$$

that takes its maximum value when $\boldsymbol{p}$ is a uniform distribution ($p_i = \frac{1}{K}$; $\forall i = 1, .., K$). This entropy measure gives the uncertainty of the outcomes of the event, but this univariate framework can be extended to situations where we are interested in the study of bidimensional distributions given by the pair of variables (*x,y*), where variable *x* can take K different values $\{x_1, x_2, ..., x_K\}$ and variable *y* can take T values $\{y_1, y_2, ..., y_T\}$. In this situation, the joint probability of a pair of random observations (x_i, y_j) will be denoted as p_{ij} and the Shannon's entropy measure for the $K \times T$ possible outcomes will be:

$$H(\boldsymbol{P}) = -\sum_{i=1}^{K}\sum_{j=1}^{T} p_{ij} ln p_{ij} \qquad (2)$$

Again, the entropy measure reaches its maximum when $\boldsymbol{P}$ is uniform. Apart from measuring the uncertainty associated to a random process, Shannon's entropy can be used for recovering an unknown probability distribution form partial or incomplete data.

We will base our explanations on the matrix-balancing problem depicted in Golan (2006, page 105), where the goal is to fill the (unknown) cells of a matrix using the information that is contained in the aggregate data of the row and column sums. Graphically, the point of departure of our problem is a matrix like Table 1.

Table 1: Known and unknown data in a matrix balancing problem.

	$z_{\cdot 1}$	...	$z_{\cdot j}$	...	$z_{\cdot T}$
$z_{1\cdot}$	z_{11}	...	z_{1j}	...	z_{1T}
...	...		...		...
$z_{i\cdot}$	z_{i1}	...	z_{ij}	...	z_{iT}
...	...		...		...
$z_{K\cdot}$	z_{K1}	...	z_{Kj}	...	z_{KT}

The z_{ij} elements of the matrix are the unknown quantities we would like to estimate, where $\sum_{j=1}^{T} z_{ij} = z_{i\cdot}$, $\sum_{i=1}^{K} z_{ij} = z_{\cdot j}$, and $\sum_{i=1}^{K}\sum_{j=1}^{T} z_{ij} = z$. These elements can be expressed as sets of (column) probability distributions, simply dividing the quantities of the matrix by the corresponding column sums, $z_{\cdot j}$. Note that. In such a case, the previous matrix can be rewritten in terms of a new matrix $\boldsymbol{P}$ that is composed by a set of *T* probability distributions (Table 2).

Table 2: The matrix balancing problem in terms of probabilities.

	y_1	...	y_j	...	y_T
x_1	p_{11}	...	p_{1j}	...	p_{1T}
...	...		...		...
x_i	p_{i1}	...	p_{ij}	...	p_{iT}
...	...		...		...
x_K	p_{K1}	...	p_{Kj}	...	p_{KT}

Where the p_{ij}'s are defined as the proportions $\frac{z_{ij}}{z_{\cdot j}}$, and the new row and column margins as $x_i = \frac{z_{i\cdot}}{z}$ and $y_j = \frac{z_{\cdot j}}{z}$ respectively. Consequently, the followings equalities are fulfilled by the p_{ij} elements[1]:

$$\sum_{i=1}^{K} p_{ij} = 1\,;\ \forall j = 1, \ldots, T \qquad (3)$$

$$\sum_{j=1}^{T} p_{ij} y_j = x_i\,;\ \forall i = 1, \ldots, K \qquad (4)$$

These two sets of equations reflect all we know about the elements of matrix $\boldsymbol{P}$. Equation (3) shows the cross-relationship between the (unknown) p_{ij}'s in the matrix and the (known) sums of each row and column. Additionally, equation (4) indicates

[1] Note that in such a case, these p_{ij} elements can be seen as conditional probabilities to each column.

that the p_{ij} *'s* can be viewed as (column) probability distributions. Note that we have only $K + T$ pieces of information to estimate the $K \times T$ elements of matrix $\boldsymbol{P}$, which makes the problem ill-posed. In such a situation, usually called a *pure linear inverse problem,* the Maximum Entropy (ME) principle can be applied to recover the unknown p_{ij} probabilities. This principle is based on the selection of the probability distribution that maximizes (5) among all the feasible probability distributions that fulfil (6) and (7). So, the following constrained maximization problem is posed:

$$\underset{\boldsymbol{P}}{\text{Max}}\, H(\boldsymbol{P}) = -\sum_{i=1}^{K}\sum_{j=1}^{T} p_{ij} ln p_{ij} \quad (5)$$

Subject to:

$$\sum_{j=1}^{T} p_{ij} y_j = x_i \,;\ \forall i = 1, \dots, K \quad (6)$$

$$\sum_{i=1}^{K} p_{ij} = 1 \,;\ \forall j = 1, \dots, T \quad (7)$$

In this problem the equations (7) are simply normalization constraints that guarantee that the estimated probabilities sum to one, and equations (6) ensure that the recovered distributions of probabilities are compatible with the aggregate data of $\boldsymbol{x}$ at all *K* observations. The Lagrangian function for such a problem will be:

$$L = -\sum_{i=1}^{K}\sum_{j=1}^{T} ln p_{ij} + \sum_{i=1}^{K} \lambda_i \left[x_i - \sum_{j=1}^{T} p_{ij} y_j \right] + \sum_{j=1}^{T} \mu_j \left[1 - \sum_{i=1}^{K} p_{ij} \right] \quad (8)$$

And the solutions (taking into account the first-order conditions) are:

$$\hat{p}_{ij} = \frac{exp[\hat{\lambda}_i y_j]}{\sum_{i=1}^{K} exp[\hat{\lambda}_i y_j]} ;\ \forall i = 1, \dots K; j = 1, \dots, T \quad (9)$$

where $\hat{\lambda}_i$ are the Lagrangian multipliers associated with constraints (6).

Alternatively to this case, it might be also possible a situation where, in addition to the information contained in the aggregate data, we have available a set of prior probabilities q_{ij}. In other words, we want to transform an *a priori* probability matrix $\boldsymbol{Q}$ into a posterior matrix $\boldsymbol{P}$ that is consistent with the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. This type of problem is frequent in some fields of economic research: for example in input-output analysis the researchers often must update an input-output matrix of coefficients to make it match with actual known row and column sums, using as *a priori* information the data collected in a previous table.

The solution to this type of problems is obtained by minimizing a divergence measure with the prior probability matrix $\boldsymbol{Q}$subject to the set of constraints (6) and (7).

The ME problem is therefore transformed into a so-called Cross-Entropy (CE) problem, which can be written in the following terms:

$$\underset{\boldsymbol{P}}{\text{Min}}\, D(\boldsymbol{P}||\boldsymbol{Q}) = \sum_{i=1}^{K}\sum_{j=1}^{T} p_{ij}\, ln\left(\frac{p_{ij}}{q_{ij}}\right) \tag{10}$$

Subject to the same restrictions given by the set of equations (6) and (7). The divergence measure $D(\boldsymbol{P}||\boldsymbol{Q})$ is the Kullback-Liebler entropy divergence between the posterior and prior distributions. The Lagrangian function for the CE problem is:

$$L = D(\boldsymbol{P}||\boldsymbol{Q}) + \sum_{i=1}^{K}\lambda_i\left[x_i - \sum_{j=1}^{T} p_{ij}y_j\right] + \sum_{j=1}^{T}\mu_j\left[1 - \sum_{i=1}^{K} p_{ij}\right] \tag{11}$$

And the solutions are:

$$\tilde{p}_{ij} = \frac{q_{ij}exp\left[\tilde{\lambda}_i y_j\right]}{\sum_{i=1}^{K} q_{ij}exp\left[\tilde{\lambda}_i y_j\right]};\ \forall i = 1, \ldots K; j = 1, \ldots, T \tag{12}$$

The CE estimation procedure can be seen as an extension of the ME principle (or alternatively the ME can be considered as a particular case of the CE procedure), given that the solutions of both approaches are the same ($\hat{p}_{ij} = \tilde{p}_{ij}$) when the T *a priori* probability distribution contained in $\boldsymbol{Q}$ are all uniform. In other words, the ME solutions are obtained by minimizing the Kullback-Liebler divergence $D(\boldsymbol{P}||\boldsymbol{Q})$ between the unknown p_{ij} and the probabilities $q_{ij} = \frac{1}{K}\ \forall i = 1, .., K$.

2.2. The ME-CE approach in the presence of noisy data.

The entropy solutions provided above to recover unknown probability distributions can be applied also to situations different from the pure inverse problems. Consider a case where, for example, the observations of vector $\boldsymbol{x}$ are “contaminated” by some measurement error; or, alternatively, a situation where the $\boldsymbol{x}$ values are affected by some uncontrolled factor other than the pure linear relationship with $\boldsymbol{y}$. In both cases, the equation (13) that relates $\boldsymbol{x}$ and $\boldsymbol{y}$ will be affected by the presence of a random disturbance $\boldsymbol{\epsilon}$ in the following terms:

$$x_i = \sum_{j=1}^{T} p_{ij}y_j + \varepsilon_i;\ \forall i = 1, \ldots, K \tag{13}$$

Or, more generally:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y} + \boldsymbol{\epsilon} \tag{14}$$

Entropy econometrics can also deal with the estimations of the unknown p_{ij} elements in such situations, which is the typical specification of a linear econometric model[2]. A first step to estimate the p_{ij} probabilities is the reparametrization of the ε_i terms, given that the CE formulation is designed for dealing with elements that behave as proper probability distributions (condition fulfilled by the p_{ij}'s but not by the ε_i's). This reparametrization allows us to generalize the use of the CE technique (Generalized Cross Entropy or GCE hereafter) to these familiar linear models.

Oppositely to other estimation techniques, GCE does not require rigid assumptions about a specific probability distribution function of the stochastic component, but it still is necessary to make some assumptions. Basically, we represent our uncertainty about the realizations of vector $\boldsymbol{\varepsilon}$ treating each element ε_i as a discrete random variable with $J \geq 2$ possible outcomes contained in a convex set $\boldsymbol{v'} = \{v_1, \dots, v_J\}$, which for the sake of simplicity is assumed as common for all the ε_i. We also assume that these possible realizations are symmetric around zero ($-v_1 = v_J$). The traditional way of fixing the upper and lower limits of this set is to apply the three-sigma rule (see Pukelsheim, 1994). Under these conditions, each element ε_i can be defined as:

$$\varepsilon_i = \sum_{h=1}^{J} w_{ih} v_h \, ; \; \forall i = 1, \dots, K \tag{15}$$

Where w_{ih} is the unknown probability of the outcome v_h for the observation *i*, which implies that $\boldsymbol{\epsilon}$ is assumed to have a mean $E[\boldsymbol{\epsilon}] = 0$ and a finite covariance matrix. From this reparametrization, equation (15) can be written as:

$$x_i = \sum_{j=1}^{T} p_{ij} y_j + \sum_{h=1}^{J} w_{ih} v_h \, ; \; \forall i = 1, \dots, K \tag{16}$$

Or, more generally:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y} + \boldsymbol{W}\boldsymbol{v} \tag{17}$$

Now we need also to estimate a $(K \times J)$ matrix $\boldsymbol{W}$ for the $(1 \times J)$ support vector $\boldsymbol{v'}$. From a matrix $\boldsymbol{W^0}$ of *a priori* probabilities, the CE program given before can be rewritten as a GCE in the following terms:

$$\underset{\boldsymbol{P},\boldsymbol{W}}{\text{Min}}\, D(\boldsymbol{P}, \boldsymbol{W} \| \boldsymbol{Q}, \boldsymbol{W^0}) = \sum_{i=1}^{K}\sum_{j=1}^{T} p_{ij} \, ln\left(\frac{p_{ij}}{q_{ij}}\right) + \sum_{i=1}^{K}\sum_{h=1}^{J} w_{ih} \, ln\left(\frac{w_{ih}}{w_{ih}^{0}}\right) \tag{18}$$

Subject to:

[2] This section will focus only on the application of the CE techniques given that, as commented before, the ME solution can be seen as a particular case of the CE approach when $q_{ij} = \frac{1}{K} \; \forall i = 1, \dots, K$.

$$x_i = \sum_{j=1}^{T} p_{ij} y_j + \sum_{h=1}^{J} w_{ih} v_h \, ; \; \forall i = 1, \dots, K \quad (19)$$

$$\sum_{i=1}^{K} p_{ij} = 1 \, ; \; \forall j = 1, \dots, T \quad (20)$$

$$\sum_{h=1}^{J} w_{ih} = 1 \, ; \; \forall i = 1, \dots, K \quad (21)$$

Note that this GCE program is the result of introducing in the pure inverse problem the estimation of the unknown probabilities $\boldsymbol{W}$corresponding to the stochastic term $\boldsymbol{\epsilon}$. The solutions of the GCE program are:

$$\tilde{p}_{ij} = \frac{q_{ij} exp\left[\tilde{\lambda}_i y_j\right]}{\sum_{i=1}^{K} q_{ij} exp\left[\tilde{\lambda}_i y_j\right]} \, ; \; \forall i = 1, \dots K; j = 1, \dots, T \quad (22)$$

$$\widetilde{w}_{ih} = \frac{exp\left[\tilde{\lambda}_i v_h\right]}{\sum_{h=1}^{J} exp\left[\tilde{\lambda}_i v_h\right]} \, ; \; \forall i = 1, \dots K; h = 1, \dots, J \quad (23)$$

Equation (22) presents an identical structure to (12) for the estimated p_{ij} probabilities. Equation (23) shows the CE solution for the estimation of w_{ih} when the *a priori* probabilities are fixed as uniform ($w_{ij}^0 = \frac{1}{J} \; \forall h = 1, .., J$), which is the natural (and most frequently applied) point of departure to reflect the high degree of uncertainty about $\boldsymbol{\epsilon}$.

2.3. Recovering individual characteristics from aggregate data: Ecological Inference based on CE-GCE techniques.

The entropy-based estimation techniques outlined in the previous subsection can be directly applied to the field of Ecological Inference (EI), which can be roughly defined as the attempt to infer individual characteristics from aggregate information. Research in this area has grown enormously in recent years, given its usefulness in many academic disciplines of social science as well as in policy analysis. The foundations of EI were introduced in the seminal works by Duncan and Davis (1953) and Goodman (1953), whose techniques were the most prominent in the field for more than forty years, although the work of King (1997) supposed a substantial development by proposing a methodology that reconciled and extended previously adopted approaches. An extensive survey of recent contributions to the field can be found in King, Rosen and Tanner (2004).

In fact, in one of the chapters of the aforementioned work, Judge et al. propose the use of information-based estimation techniques in the field of EI, although their proposal is made in a different context (the estimation of individual voters' behavior from aggregate election data). Peeters and Chasco (2006) also combined entropy econometrics in the context with EI, but in a different way to that proposed in this paper. Roughly speaking, they used GCE to estimate a weighted regression model that allows for recovering characteristics at a regional scale from information at a national level.

To explain how the GCE technique can be applied in the context of EI, consider a geographical area (a country, for example) that can be divided in *T* smaller spatial units (regions). Besides this first geographical partition, suppose that another division is also possible in accordance with another characteristic. Consider that the second criterion applied for this additional partition is a classification of the municipalities that make up the country, obtaining *K* different types of municipalities. Within such a context, the objective would be to estimate how a variable is distributed among the regions according to the classification of municipalities, using aggregate data as information. Graphically, this estimation problem can be represented by a grid with the same structure as Table 2.

Table 3: A spatial division across regions and type of municipality.

		Regions				
		y_1	.	y_j	.	y_T
Type of municipality	x_1	p_{11}	.	p_{1j}	.	p_{1T}
	.	.		.		.
	x_i	p_{i1}	.	p_{ij}	.	p_{iT}
	.	.		.		.
	x_K	p_{K1}	.	p_{Kj}	.	p_{KT}

Each one of the p_{ij}*'s* is now defined as the (unknown) proportion of the variable that is allocated in the municipalities of type *i* situated in the region *j*, forming a $(K \times T)$ matrix ***P*** with *T* unknown probability distributions. The $(1 \times T)$ row vector ***y*** represents the regional proportions of the variable and the $(K \times 1)$ column vector ***x*** shows the national allocation of the variable according to the type of municipality. Note that these two vectors contain the aggregate data existing for the researcher, which our EI estimation will be based on. If an *a priori* set of probability distributions ***Q*** is also available, the cross entropy procedures outlined previously can be directly applied.

Note that both the CE technique for pure inverse problem as well as a GCE program that include the presence of a random term are applicable in this context, and it is a decision to be made by the researcher to follow one specific approach. In the first case, we will assume that there is a pure linear relationship between the row and column margins of our matrix, and the following CE program would have to be solved:

$$\underset{\boldsymbol{P}}{\text{Min}}\, D(\boldsymbol{P}\|\boldsymbol{Q}) \quad (24)$$

Subject to:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}' \quad (25)$$

$$\boldsymbol{e}'_{\boldsymbol{K}}\boldsymbol{P} = \boldsymbol{e}'_{\boldsymbol{K}} \quad (26)$$

Where $\boldsymbol{e_K}$ stands for an appropriate (column) vector of ones. Alternatively, if it seems realistic the inclusion of a random term that affects the observations of vector $\boldsymbol{x}$, it would be necessary to solve the following GCE program and estimate jointly matrices $\boldsymbol{P}$ and $\boldsymbol{W}$:

$$\underset{\boldsymbol{P},\boldsymbol{W}}{\text{Min}}\, D(\boldsymbol{P},\boldsymbol{W}\|\boldsymbol{Q},\boldsymbol{W^0}) \quad (27)$$

Subject to:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}' + \boldsymbol{W}\boldsymbol{v} \quad (28)$$

$$\boldsymbol{e}'_{\boldsymbol{K}}\boldsymbol{P} = \boldsymbol{e}'_{\boldsymbol{K}} \quad (29)$$

$$\boldsymbol{W}\boldsymbol{e_J} = \boldsymbol{e_J} \quad (30)$$

Being $\boldsymbol{e_J}$ the corresponding column vector of ones.

3. Estimating 2001 urban income in Spain according to city size.

3.1. Estimation procedure.

Spanish official data on income at a municipal level are not generally available (, so an estimation procedure is necessary.

Spanish municipalities can be posed in similar terms to the matrix balancing problems described in previous sections. Spain is administratively divided in 50 provinces for which data on income is available in the Regional Accounts annually elaborated by the Spanish Statistical Institute (INE). Additionally, from 1998 to 2004 the INE also produced the Continuous Survey on Household Budgets (ECPF), where one can find information of income and expenditure characteristics from a quarterly sample of approximately 8.000 Spanish families[3]. Particularly interesting for our research, the longitudinal files containing the microdata provide annual information about the personal income distribution according to the type of municipality where the household lived at the time of being surveyed. Specifically, this municipal classification is as appear in Table 4.

3 More detailed information on these surveys can be found in www.ine.es.

Table 4: Classification on the Spanish municipalities on the Continuous Survey on Household Budgets.

Type of municipality	Description
m_1	Capital city of the province (independently on its population)
m_2	Municipality with more than 100,000 inhabitants
m_3	Municipality with a population between 50,000 and 100,000
m_4	Municipality with a population between 20,000 and 50,000
m_5	Municipality with a population between 10,000 and 20,000
m_6	Municipality with less than 10,000 inhabitants

Note that this partition of the Spanish municipalities does not correspond exactly with the population size given that the category “capital city” does not reflect exactly the population size. Even so, this classification can be seen as a good indicator of the spatial distribution of income according to the size of the municipalities, given that there is a little number of provinces (Asturias, Cadiz, Pontevedra and Toledo are the only exceptions) where the capital is smaller than some other city on the same province.

The information sources described above allow for obtaining the row and column margins represented by the vectors ***x*** and ***y*** in Table 3. Vector ***x***, with dimension (6 × 1), contains the proportion of income per type of municipality and the (1 × 50) vector ***y*** with the provincial proportions of income. From these aggregate data, we will apply the entropy-based estimation strategies explained in previous sections to recover the allocation of provincial income according to the type of municipality for 2001. We have chosen this specific year because this is also the reference year of the most recent census elaborated in Spain[4], which provides information for specifying a natural *a priori* distribution $\boldsymbol{Q}$ based on the provincial distribution of labor per type of municipality. From this point of departure, two parallel estimation procedures have been applied.

Let us first assume that we can pose a pure linear relationship between vectors ***x*** and ***y*** to solve the following CE problem:

$$\underset{\boldsymbol{P}}{\text{Min}}\, D(\boldsymbol{P}\|\boldsymbol{Q}) = \sum_{i=1}^{6}\sum_{j=1}^{50} p_{ij}\, ln\left(\frac{p_{ij}}{q_{ij}}\right) \quad (31)$$

Subject to:

$$x_i = \sum_{j=1}^{T} p_{ij} y_j\,;\ \forall i = 1, \dots, 6 \quad (32)$$

[4] For details about the Spanish Census, see http://www.ine.es/censo2001/infotec.htm.

$$\sum_{i=1}^{K} p_{ij} = 1 \,;\ \forall j = 1, \dots, 50 \qquad (33)$$

The solution to this CE program is reported in Table 5 for all the Spanish provinces. The income values have been obtained as the respective estimate of p_{ij} multiplied by the total income of province *j*. Note also that, instead of showing the income value, the estimates have been divided by the respective population sizes to provide results of income per capita (in thousands of Euros). The last column of the table, which is shaded in grey, shows the proportions of income per province. Similarly, the last row is shaded in grey as well, and contains the proportion of income per type of municipality. Note that these proportions correspond to vectors $\boldsymbol{y}$ and $\boldsymbol{x}$ respectively, the aggregate information used for the estimation, although they have been transposed from their usual position in previous sections in order to fit the tables into the size of the pages.

Another possibility is to include a stochastic term in the linear model that relates $\boldsymbol{x}$ and $\boldsymbol{y}$ and transform the pure linear inverse relationship in a more general linear model. This can be justified by the fact that the observations of the proportions are obtained from samples, which implies the possibility that these observations might have been affected by some measurement error. In general, it may be unrealistic to assume that the $\boldsymbol{x}$ and $\boldsymbol{y}$ vectors are perfectly observed, so it seems plausible to consider a model like (17) with both systematic and stochastic components. It turns the CE problem into the following GCE program:

$$\underset{\boldsymbol{P},\boldsymbol{W}}{\text{Min}}\, D(\boldsymbol{P},\boldsymbol{W}\|\boldsymbol{Q},\boldsymbol{W^0}) = \sum_{i=1}^{6}\sum_{j=1}^{50} p_{ij}\, ln\left(\frac{p_{ij}}{q_{ij}}\right) + \sum_{i=1}^{6}\sum_{h=1}^{3} w_{ih}\, ln\left(\frac{w_{ih}}{w_{ih}^{0}}\right) \qquad (34)$$

Subject to:

$$x_i = \sum_{j=1}^{T} p_{ij} y_j + \sum_{h=1}^{3} w_{ih} v_h \,;\ \forall i = 1, \dots, 6 \qquad (35)$$

$$\sum_{i=1}^{K} p_{ij} = 1 \,;\ \forall j = 1, \dots, 50 \qquad (36)$$

$$\sum_{h=1}^{3} w_{ih} = 1 \,;\ \forall i = 1, \dots, 6 \qquad (37)$$

The support vector $\boldsymbol{v'}$ contains the possible values and it has $J = 3$ elements centred on 0 and is defined as $\boldsymbol{v}' = [-\alpha, 0, \alpha]$. If we had perfect knowledge of the variability present on $\boldsymbol{x}$, a reasonable rule for α is the three-standard deviation rule (Pukelsheim, 1994). However, given our incomplete knowledge, we will follow the proposal made in Golan et al. (1997), where the entropy econometrics techniques are applied to linear models with multinomial data on the dependent variable, and we will use the sample variance of $\boldsymbol{x}$ as an estimate for α. The a priori probabilities $\boldsymbol{W}^{\mathbf{0}}$ for the error have been fixed as uniform, as explained before. Table 6, which presents the same structure of rows and columns as Table 5, shows the solutions to this GCE estimation problem[5].

Note that the estimates reported on both tables are very similar, which suggests that the outcomes are relatively robust to changes in the specification of the estimation procedures.

3.2. Discussion of the results.

The results obtained for the Spanish case fit basic theoretical economic expectations. The highest estimates of GDP's per capita are obtained for large urban areas, confirming the relevance of agglomeration economies: the bigger the city, the larger the GDP per capita. This is especially clear for the largest cities in the country. The differences between the two main metropolises (Madrid and Barcelona) and the rest of the municipalities, including other large cities, are noteworthy. Small cities and rural areas present lower GDP's per capita with some exceptions for places located very close to a large metropolis.

To be able to interpret and analyse these results, we can apply a typical classification of territory to these data based on the approaches of Coffey and Polèse (1988), Polèse and Champagne (1999), Polèse and Shearmur (2004) and Polèse, Rubiera and Shearmur (2007), which take into account *size* and *distance* effects (possibly the two most important effects in business localization and regional growth). First, we can distinguish all the spaces that can be considered large metropolises. All the aforementioned studies show a strong trend of higher growth rates, particularly in strategic economic sectors such as knowledge-intensive business services, in and around cities and, more specifically, in and around large metropolitan areas. We can then classify the remaining territories in terms of their distance to a large metropolis as central or peripheral areas. Central and peripheral areas could also be classified according to their size. As a result, we have five types of regions: (i) Metropolitan Areas (MA), (ii) Urban Central Areas (UCA), (iii) Urban Peripheral Areas (UPA), (iv) Rural Central Areas (RCA) and (v) Rural Peripheral Areas (RPA), taking in account the fact that UCA and UPA could be classified in different types according to their sizes.

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[5] The blank cells in both tables correspond with a type of municipalities that does not exist in a specific province.

Table 5: CE estimates of income per type of municipality.

(thousands €/person)

	m_1	m_2	m_3	m_4	m_5	m_6	y
Almeria	14.36		18.22	16.13	16.12	13.77	0.0123
Cádiz	13.68	12.84	11.85	11.98	12.42	10.44	0.0207
Cordoba	11.42			10.83	10.75	9.67	0.0122
Granada	12.09		12.16	8.09	12.78	10.36	0.0134
Huelva	14.00			12.55	13.18	11.81	0.0089
Jaen	12.79		9.75	10.52	10.93	10.06	0.0103
Málaga	13.19	12.74	5.84	13.84	11.19	10.71	0.0240
Sevilla	16.24	12.20	11.21	7.19	10.56	10.52	0.0319
Huesca	19.86				14.09	17.48	0.0053
Teruel	20.34				19.78	15.51	0.0035
Zaragoza	17.92				16.02	15.55	0.0222
Asturias	16.75	14.26	13.36	11.88	12.76	13.76	0.0221
Baleares	21.22			16.36	16.89	20.17	0.0258
Las Palmas	15.89		14.76	15.95	14.76	18.33	0.0222
Tenerife	14.21	14.61	14.68	12.58	15.68	14.62	0.0186
Cantabria	15.87		14.34	16.39	15.46	15.69	0.0125
Avila	14.90					11.87	0.0031
Burgos	18.64			17.19		17.25	0.0093
León	14.89		13.40	16.04	9.59	13.64	0.0100
Palencia	15.63					14.51	0.0039
Salamanca	14.59				13.34	12.63	0.0070
Segovia	17.01					15.48	0.0035
Soria	16.89					15.18	0.0021
Valladolid	16.95			14.83	18.98	16.39	0.0124
Zamora	13.23				12.61	10.81	0.0035
Albacete	13.41			12.29	12.16	10.84	0.0067
Ciudad Real	15.11		10.95	13.81	13.63	12.25	0.0093
Cuenca	13.99				13.53	11.84	0.0037
Guadalajara	16.04			17.31		12.51	0.0038
Toledo	14.75		12.42		17.13	12.08	0.0104
Barcelona	28.26	16.35	13.12	13.13	13.94	22.43	0.1423
Girona	18.36			18.28	19.11	20.70	0.0173
Lleida	20.86				19.95	19.75	0.0110
Tarragona	22.10		19.71	19.96	19.71	20.66	0.0191
Alicante	16.44	14.71	11.83	10.55	15.96	16.50	0.0322
Castellon	19.14			18.00	18.80	17.55	0.0135
Valencia	17.65		13.29	12.96	15.84	15.98	0.0523
Badajoz	12.65		12.27	10.96	10.90	9.21	0.0103
Cáceres	11.90			10.73	7.92	10.44	0.0064
Coruña	14.46		12.07	13.31	12.39	12.95	0.0215
Lugo	14.19				12.37	11.56	0.0066
Orense	13.49				12.43	10.68	0.0060
Pontevedra	13.80	12.98		10.90	14.14	12.60	0.0175
Madrid	28.74	14.15	11.73	10.63	13.45	18.05	0.1779
Murcia	14.96	12.19	13.90	13.13	12.96	12.24	0.0244
Navarra	22.18			20.50	16.10	20.07	0.0172
Alava	23.02				18.90	21.55	0.0097
Guipúzcoa	22.09		20.00	20.30	20.40	21.97	0.0213
Vizcaya	20.59		16.76	17.73	19.33	20.23	0.0318

La Rioja	18.96			17.74	17.42	17.39	0.0076
x'	0.4244	0.0804	0.0679	0.1159	0.1000	0.2115	

Table 6: GCE estimates of income per type of municipality.

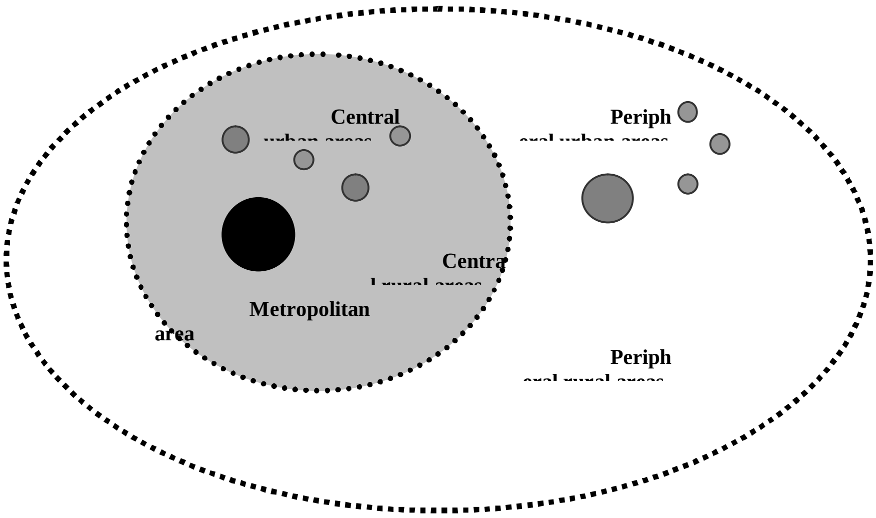
(thousands €/person)

	m_1	m_2	m_3	m_4	m_5	m_6	y
Almeria	14.34		18.28	16.17	16.13	13.74	0.0123
Cádiz	13.63	12.81	11.89	12.01	12.42	10.40	0.0207
Cordoba	11.41			10.86	10.77	9.66	0.0122
Granada	12.07		12.21	8.12	12.81	10.35	0.0134
Huelva	13.99			12.58	13.20	11.81	0.0089
Jaen	12.78		9.78	10.54	10.95	10.04	0.0103
Málaga	13.15	12.74	5.87	13.90	11.21	10.68	0.0240
Sevilla	16.20	12.20	11.31	7.24	10.61	10.49	0.0319
Huesca	19.85				14.10	17.48	0.0053
Teruel	20.34				19.79	15.51	0.0035
Zaragoza	17.91				16.09	15.55	0.0222
Asturias	16.70	14.24	13.43	11.93	12.79	13.72	0.0221
Baleares	21.16			16.45	16.94	20.11	0.0258
Las Palmas	15.84		14.83	16.01	14.78	18.26	0.0222
Tenerife	14.17	14.60	14.75	12.62	15.71	14.58	0.0186
Cantabria	15.85		14.39	16.44	15.48	15.67	0.0125
Avila	14.90					11.87	0.0031
Burgos	18.63			17.23		17.24	0.0093
León	14.88		13.44	16.08	9.60	13.63	0.0100
Palencia	15.63					14.51	0.0039
Salamanca	14.59				13.36	12.63	0.0070
Segovia	17.01					15.48	0.0035
Soria	16.89					15.18	0.0021
Valladolid	16.94			14.88	19.02	16.39	0.0124
Zamora	13.23				12.62	10.81	0.0035
Albacete	13.40			12.31	12.17	10.83	0.0067
Ciudad Real	15.09		10.98	13.84	13.64	12.23	0.0093
Cuenca	13.99				13.54	11.84	0.0037
Guadalajara	16.04			17.32		12.51	0.0038
Toledo	14.74		12.46		17.16	12.07	0.0104
Barcelona	27.89	16.35	13.64	13.54	14.19	22.13	0.1423
Girona	18.33			18.35	19.15	20.67	0.0173
Lleida	20.85				19.99	19.75	0.0110
Tarragona	22.05		19.80	20.03	19.75	20.61	0.0191
Alicante	16.36	14.68	11.91	10.60	15.99	16.42	0.0322
Castellon	19.11			18.04	18.83	17.52	0.0135
Valencia	17.54		13.46	13.09	15.92	15.88	0.0523
Badajoz	12.64		12.30	10.99	10.92	9.20	0.0103
Cáceres	11.90			10.74	7.93	10.44	0.0064
Coruña	14.42		12.13	13.36	12.41	12.91	0.0215
Lugo	14.19				12.38	11.56	0.0066
Orense	13.48				12.45	10.68	0.0060
Pontevedra	13.77	12.98		10.93	14.16	12.57	0.0175
Madrid	28.50	14.27	12.42	11.15	13.87	17.90	0.1779
Murcia	14.91	12.17	13.98	13.18	12.99	12.19	0.0244
Navarra	22.17			20.59	16.15	20.05	0.0172
Alava	23.02				18.94	21.55	0.0097

Guipúzcoa	22.03		20.10	20.38	20.44	21.90	0.0213
Vizcaya	20.49		16.87	17.82	19.37	20.13	0.0318
La Rioja	18.96			17.78	17.45	17.39	0.0076
x'	0.4244	0.0804	0.0679	0.1159	0.1000	0.2115	

Figure 1 presents a schematic representation for an idealized national space economy. The reader will undoubtedly note the resemblance with the classic idealized economic landscapes of Christaller, Lösch, and Von Thünen, all of which posit one metropolis or marketplace at the centre. Figure 1 shows one metropolis at the centre, but also other smaller “central” urban areas of different population sizes (urban areas close to the metropolis) as well as “central” rural areas (close to the metropolis). Other analogous territories are labelled as “peripheral” urban areas, located at some distance from the metropolis, surrounded by corresponding rural places. It is implicitly assumed that urban areas are distributed in accordance with the rank-size rule.

Figure 1: Schematic Representation of the Classification of Spatial Units.



The results of Tables 5 and 6 have been plotted in Figure 2, focusing only on the GCE estimates[6]. Madrid and Barcelona constitute Type-1 Metropolitan Areas (MA1), while Type-2 Metropolitan Areas (MA2) are made up of the remaining large cities with more than five hundred thousand inhabitants. These cities were identified because it is possible to identify the province and the types of municipalities in Tables 5 and 6. The rest of the municipalities are classified in UA1 and UA2. UA1 cities have a population of more than one hundred thousand inhabitants, while UA2 cities have more than ten thousand inhabitants, but less than one hundred thousand. Finally rural areas (municipalities with less than ten thousand inhabitants) are labeled as RA. The solid line represents the values of the areas located close (less than one hour’s drive) to a large metropolis (MA1 or MA2) and are labeled as central areas (CUA1, CUA2 and CRA). The dotted line represents the values of the cities located far away (approximately more than one hour’s drive[7]) and are labeled as peripheral areas (PUA1, PUA2 and PRA).

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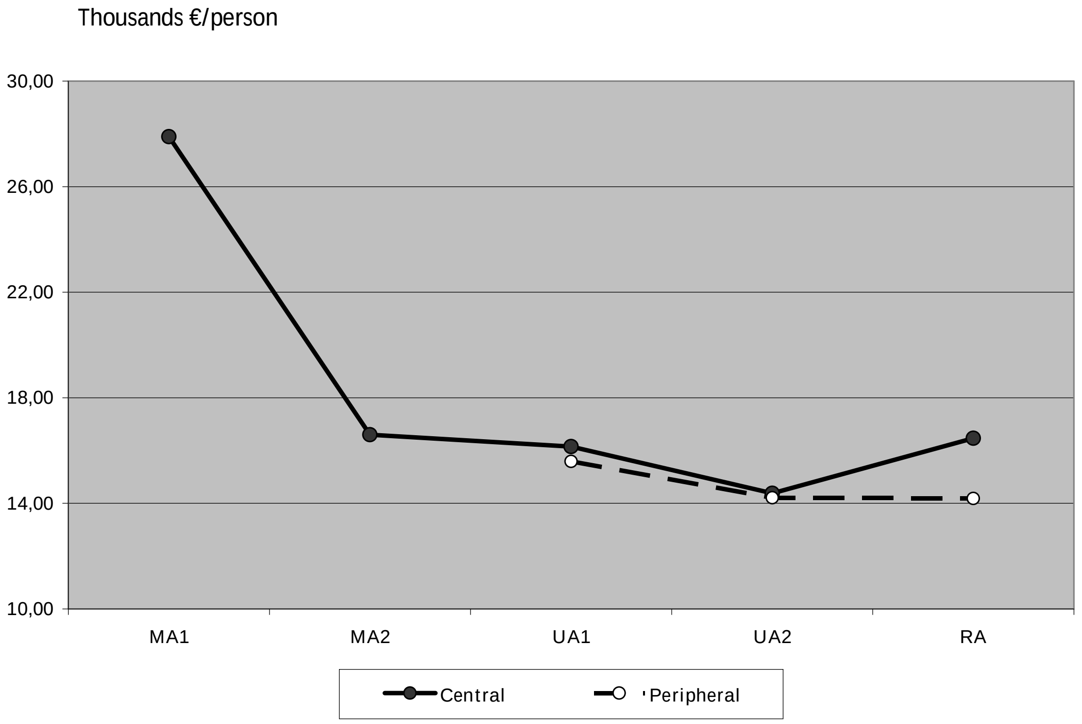
[6] Results for the CE estimates are very similar; any relevant conclusion does not change.

[7] A delimitation of all the Spanish territory as central or peripheral was made at a local level in Polèse, Rubiera and Shearmur (2007). Most of the municipalities can be classified knowing only the size and province in which they are located. Although this is an approximation due to some cases being miss-classified, they are however urban areas with

It seems that the most relevant factor to explain spatial differences in GDP per capita is the size of the main urban center of the territory. The main metropolises present the highest values by far (approximately twice the average of the rest of the country). The differences between large and medium-sized cities are less relevant, while being located close to or far from a large metropolis is found to be almost irrelevant. The continuous line is always above the dotted one. This indicates that being a medium-sized city located close to a large metropolis permits the attraction of part of the growth of the metropolis due to the expulsion of space intensive activities such as manufacturing plants from large cities. However, the differences explained by distance are especially clear in rural areas. Peripheral rural areas have a significantly lower GDP per capita than central rural areas, given that the latter are usually residential areas with many commercial services, while the former are based on agricultural activities.
These first results confirm the relevance of agglomeration economies not only for large cities, but also for medium-sized and small size cities and even rural areas located close to a large metropolis. Nevertheless, more precise analyses over an extended time period are needed to confirm and estimate the actual role of size and distance. These analyses could be possible with the extension of the methodology proposed in this paper to estimate the GDP over several years and other relevant variables in empirical regional studies.

Figure 2: Representation of average GCE estimates of income per type of municipality.



a relevant size. Consequently, no fundamental change is expected in the basic results due to these cases that we are not able to identify clearly.

4. Conclusions and Future Research.

One of the main problems of regional studies is related to the difficulty of working with spatial classifications with economic sense. Normally, data are not available at a local level and economic researchers must deal with databases structured according to political or administrative criteria. Although certain aggregations of the information are possible, these do not usually suffice to construct economically meaningful sets of regions.

This paper proposes a methodology based on Entropy Econometrics to estimate data at a local level according to the size of each basic spatial unit. This estimation exercise allows the inference of information of regions grouped according to a clear economic criterion: urban size. In particular, we propose a specific classification that jointly considers urban size and distance from the main metropolis. This allows the measurement of agglomeration economics and location effects.

The methodology proposed is applied to Spain, obtaining data at a local level of GDP per capita for the year 2001. The results obtained are in line with previous work by other authors with respect to particular cases (see, for example, Chasco (2003) and Chasco and López (2004)): the larger the city, the higher the local GDP per capita. The size effect is especially clear in the biggest cities, Madrid and Barcelona. Position likewise seems quite relevant. Cities located close to a large metropolis present higher GDP's per person than those located far away. The same also holds for the case of rural areas.

The methodology was tested using a Monte Carlo simulation experiment which concludes that this way of estimating local data is reasonably reliable.

The proposed methodology opens up broad possibilities that could be explored in subsequent research. The focus of this paper was to propose the methodology and test its possibilities with a real world example. Including a temporal dimension, where this estimation exercise could be carried out over several years, would allow not only the estimation of GDP differences, but also the evolution and growth of the different areas. Convergence analysis would also be possible using different regional classifications. This would be useful to test whether the convergence trends identified for Spanish regions are maintained when regions are constructed in terms of an economic and not an administrative criterion.

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Appendix: Testing the results by a numerical experiment.

Although the general properties of the CE-GCE estimators have been largely studied in the literature (see for example Golan et al., 1996, or Golan, 2006), some doubts about the accuracy of the specific estimates reported in the paper might emerge. In order to test if the entropy-based techniques applied in the section 5 of the paper perform well in such conditions, a simple numerical experiment has been carried out. The goal of this exercise is to get some empirical evidence on the performance of the CE and CGE approaches to estimate a unknown (6 × 50) matrix $\boldsymbol{P}$ of probabilities from aggregate data and some *a priori* matrix $\boldsymbol{Q}$.

Our Monte Carlo experiment will start out from the actual vector $\boldsymbol{y}$ of proportions of income for the Spanish provinces in 2001 and it is kept fixed throughout the simulations. Additionally, a randomly generated matrix $\boldsymbol{P}$ is obtained in each trial of the simulation; this matrix is composed by elements p_{ij} that have been drawn from a uniform distribution $p_{ij} \sim U[0{,}0.2]$; $i = 1, \dots, 5$; and $p_{6j} = 1 - \sum_{i=1}^{5} p_{ij}$ in order to assure that they behave as a set of proper (column) probability distributions. Based on the linear relationship $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}'$, vector $\boldsymbol{x}$ is obtained in each trial, and together with the observations of vector $\boldsymbol{y}$, it represents the aggregate data to obtain the estimates of the (now assumed) unknown matrix $\boldsymbol{P}$. Another important piece in the estimation process is the choice of the matrix $\boldsymbol{Q}$. To reflect the idea that the specification of this *a priori* matrix can be more or less similar to the matrix $\boldsymbol{P}$, in our experiment the cells of $\boldsymbol{Q}$ have been generated from $\boldsymbol{P}$ and a random disturbance $\boldsymbol{u}$ in the following way[8]:

$$\left.\begin{aligned} q_{ij} &= (p_{ij}) \cdot (u_{ij});\ \forall i = 1, \dots, 5;\ \forall j = 1, \dots, 50 \\ q_{6j} &= 1 - \sum_{i=1}^{5} p_{ij}\,;\ \ \forall j = 1, \dots, 50 \end{aligned}\right\} \qquad (38)$$

Where $\boldsymbol{u} \sim N(1, \sigma)$ and σ is a scalar. Note that if $= 0$, then $p_{ij} = q_{ij}$ for all the cells of both matrices. The bigger the value of σ, the larger the divergence between matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ and hence the smaller the expected accuracy of the estimation. This consequence is rather logical, given that a good specification of the $\boldsymbol{Q}$ matrix (close to the real $\boldsymbol{P}$ matrix) will be helpful in the estimation process. On the contrary, if the chosen $\boldsymbol{Q}$ differs significantly from the actual $\boldsymbol{P}$, the data observed in the sample (the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$) will have more difficulties to lead the estimates to solutions close to the real values.

In the experiment six different scenarios have been simulated for several values of the scalar σ: 0.1, 0.2, 0.25, 0.35, 0.4 and 0.5. Both the CE and the GCE (applying in this last case the three-sigma rule for the support of the error term) solutions have been obtained under these levels of divergence between $\boldsymbol{P}$ and $\boldsymbol{Q}$. In each one of these six scenarios 1,000 trials have been carried out and the average of two overall

[8] This approach is based on the experiment carried out in Golan et al. (1996, pages 63 and 64). To avoid undesirable negative values on q_{ij} $\forall i = 1, \dots, 5$; if the number generation obtained a negative, it has been replaced by $q_{ij} = 10^{-8}$.

measures of error have been computed: the root of the mean squared error (RMSE), which was obtained as $RMSE = \sqrt{\frac{1}{50\times 6}\sum_{i=1}^{6}\sum_{j=1}^{50}\left(\tilde{p}_{ij} - p_{ij}\right)^2}$, and the mean absolute error (MAE), defined as $MAE = \frac{1}{50\times 6}\sum_{i=1}^{6}\sum_{j=1}^{50}\left|\left(\tilde{p}_{ij} - p_{ij}\right)\right|$, where $\tilde{p}_{ij}$ stands for both the CE and GCE estimates. The Table A1 shows the results of these error measures.

Table A1: Error measures in the Monte Carlo simulation.

CE estimation	σ = 0.5	σ = 0.4	σ = 0.35	σ = 0.25	σ = 0.2	σ = 0.1
RMSE	0.005	0.003	0.003	0.001	0.001	0.000
MAE	0.049	0.040	0.035	0.025	0.020	0.010
GCE estimation	σ = 0.5	σ = 0.4	σ = 0.35	σ = 0.25	σ = 0.2	σ = 0.1
RMSE	0.072	0.059	0.052	0.037	0.030	0.015
MAE	0.050	0.040	0.036	0.026	0.021	0.010

As expected, the error measure are (slightly) larger in all cases if we apply a GCE estimation program compared with the estimates obtained a CE approach. This result is not surprising, given that the GCE allows for the presence of an error term that prevents an exact match between the row and column margins through the estimate of matrix $\boldsymbol{P}$. Moreover, the deviations between real and estimated p_{ij} elements increase as the divergence between the a priori $\boldsymbol{Q}$ and the real matrix $\boldsymbol{P}$ get bigger. Although the RMSE measure seems more sensitive to the specification choice between a pure CE or a GCE estimation program, both error measures, RMSE and MAE, remained at moderate levels even for considerably big values of the scalar σ.

These outcomes give a rough idea on the size of the error that presumably our empirical application on section 5 can present. If we compare the distribution of income per province with the provincial distribution of labor in the census (both taken in 2001) by means of a quotient, which is similar to the $\boldsymbol{u}$ disturbance considered in the Monte Carlo experiment, we obtain a (50 × 1) vector that behaves approximately as a normal distribution and with a sample standard deviation of 0.19. This result suggests that the estimates obtained for the local per capita income, based on the estimates of the unknown p_{ij} elements, can be taken as reasonably reliable for the case of Spain.